

Mirror Transitions and the Batyrev-Borisov Construction

Karl Fredrickson

Abstract

We consider examples of extremal transitions between families of Calabi-Yau complete intersection threefolds in toric varieties, which are induced by toric embeddings of one ambient toric variety into the other. We show that the toric map induced by the linear dual of the embedding induces a birational morphism between the mirror Calabi-Yau families, and give evidence that it can be extended to the full mirror transition between mirror families.

1 Introduction

Suppose that X and Y are smooth Calabi-Yau threefolds, which we will also take to be projective algebraic varieties over \mathbb{C} . We say that X and Y are related by a *conifold transition* if there exists a degeneration of X to a singular variety X_0 with a finite number of conifold singularities, and Y can be obtained from X_0 by a small resolution of singularities. Here, a conifold singularity is a certain type of isolated singularity (which includes the case of ordinary double points), and a small resolution replaces each singularity with a union of projective lines in such a way that the resolved variety has a trivial canonical bundle. There is a conjecture that all pairs of Calabi-Yau threefolds can be obtained from one another by a finite number of such transitions; see the paper [9]. More generally, one can consider the idea of an *extremal transition* of Calabi-Yau threefolds, defined in [7], where Y is still obtained from X by degenerating and resolving, but the singularities on X_0 may be much worse than conifold singularities.

The main example in this paper will focus on a case where the conifold singularities are ordinary double points. An ordinary double point, which we

will also refer to as a node, is a singularity which is locally isomorphic, up to a complex analytic change of coordinates, to the singularity of the affine variety $A \subseteq \operatorname{Spec} \mathbb{C}[x, y, z, w]$ defined by $xy - zw = 0$. A small resolution of an ordinary double point replaces the singularity with a \mathbb{P}^1 , and is locally isomorphic, again up to a complex analytic change of coordinates, to the resolution of the singularity of A given by blowing up the ambient space \mathbb{C}^4 on the plane $x = z = 0$.

The paper [7] proposes an elegant connection between extremal transitions and the concept of mirror symmetry introduced by physicists studying string theory. If X and Y are related by an extremal transition, and X^* and Y^* are their mirrors, then Morrison proposes that X^* and Y^* are also related by an extremal transition, but with the degeneration and resolution switched. In other words, there should be a degeneration of Y^* to a singular variety Y_0^* , and X^* should be a resolution of Y_0^* . In this paper, we study a particular example of a conifold transition and propose that the “mirror” conifold transition can be realized by a toric map between the ambient toric varieties of the mirror Calabi-Yau threefolds as provided by the Batyrev-Borisov construction. The Batyrev-Borisov construction, described in [2], is the fundamental method for constructing mirrors of Calabi-Yau manifolds which are complete intersections in Gorenstein toric Fano varieties.

The relationship between extremal transitions and the Batyrev-Borisov construction has also been explored by Mavlyutov in [6], where he gives a construction based on deforming a Calabi-Yau hypersurface in a toric variety by embedding the toric variety as a complete intersection in a higher dimensional toric variety. This is exactly how the deformations arise in both of the examples in this paper, although Mavlyutov’s methods seem to require Minkowski sum decompositions of reflexive polytopes that do not exist in our situations.

In [7], Morrison himself discusses the relationship between extremal transitions and Batyrev’s construction in [1], which is really just the Batyrev-Borisov construction in the case of hypersurfaces. One may consider a simple inclusion $P \subseteq Q$ of reflexive polytopes as an extremal transition between Calabi-Yau hypersurfaces in the toric varieties corresponding to P and Q . According to Batyrev’s construction, the mirror hypersurfaces will lie in the toric varieties corresponding to the dual polytopes P^* and Q^* , which will obey the reverse inclusion $Q^* \subseteq P^*$. This induces an extremal transition between the mirror hypersurfaces. However, this simpler technique cannot be used in the examples we will look at, since we will deal with a Calabi-Yau

manifold which is a codimension two complete intersection in a toric variety, rather than a hypersurface.

Conifold transitions were used by Batyrev et al. in [3] to construct mirrors of complete intersection Calabi-Yau threefolds in Grassmannians. The main example in this paper are related to the case of quartic hypersurfaces in the Grassmannian $G(2, 4)$. The approach of Batyrev et al. is to degenerate the Grassmannian $G(k, n)$ to a Gorenstein toric Fano variety called $P(k, n)$. After the degeneration, a family of complete intersection Calabi-Yau threefolds in $G(k, n)$, \mathcal{X} , will degenerate to a family of complete intersections \mathcal{X}_0 in $P(k, n)$ where the general member of the family has isolated nodes as singularities. A so-called MPCP resolution of $P(k, n)$ (as defined in [1]) gives a small resolution of general members of \mathcal{X}_0 , thus giving a nonsingular family $\widehat{\mathcal{X}}_0$ which is related by a conifold transition to the original family \mathcal{X} . Since $\widehat{\mathcal{X}}_0$ consists of MPCP resolutions of complete intersections in the Gorenstein toric Fano variety $P(k, n)$, the Batyrev-Borisov construction can be applied directly to the family $\widehat{\mathcal{X}}_0$ to yield the mirror family $\widehat{\mathcal{X}}_0^*$. By Morrison's philosophy, the mirror family to the original family \mathcal{X} should be produced by a mirror conifold transition of $\widehat{\mathcal{X}}_0^*$. That is, we should obtain the mirror family to \mathcal{X} after an appropriate degeneration of $\widehat{\mathcal{X}}_0^*$ to a family where the general member has nodes, followed by a small resolution.

In [3] this degeneration is accomplished by imposing a relation on the coefficients in the equations describing the family, so the degenerate family is actually a subfamily of $\widehat{\mathcal{X}}_0^*$. In the case of quartic hypersurfaces in $G(2, 4)$, $\widehat{\mathcal{X}}_0^*$ is contained in an MPCP resolution of a toric variety $X(\Delta_{P(2,4)}^*)$ (the toric variety given by taking cones over the Newton polytope of quartics on $P(2, 4)$, which is reflexive). We will refer to the degenerate nodal subfamily of $\widehat{\mathcal{X}}_0^*$ in the $G(2, 4)$ case, which consists of hypersurfaces in an MPCP resolution of $X(\Delta_{P(2,4)}^*)$, as \mathcal{X}_C^* .

Quartic hypersurfaces in $G(2, 4)$ are also smooth members of the family $\mathcal{X}_{(2,4)}$ of $(2, 4)$ complete intersections in \mathbb{P}^5 , since $G(2, 4)$ is defined by a single quadratic equation in \mathbb{P}^5 . Being complete intersections in the Gorenstein toric Fano variety \mathbb{P}^5 , the Batyrev-Borisov construction also provides a way to find the mirror of this family of Calabi-Yau threefolds. Thus there are two possible methods that may be used to find the mirror family of quartic hypersurfaces in $G(2, 4)$. Let us denote by \mathcal{X}_{BB}^* the mirror family given by the Batyrev-Borisov construction.

We start by establishing a birational morphism between the families \mathcal{X}_C^*

and \mathcal{X}_{BB}^* . This birational morphism is realized by a toric (monomial) map between the open tori of the toric varieties containing the families. Furthermore, as a linear map between fans, this map is just the linear dual to the toric embedding of $P(2, 4)$ into \mathbb{P}^5 as the zero locus of a quadratic binomial equation.

This birational morphism will only define a map between open subsets of members of the two Calabi-Yau families, so to remedy this, we show that the map can be extended to a map between larger toric varieties. Although the domain of the map will not be the entire ambient toric variety of \mathcal{X}_{BB}^* provided by the Batyrev-Borisov construction (i.e. it will not be compact), it will be large enough to contain generic members of \mathcal{X}_{BB}^* . We conjecture that this map induces a small resolution of the family \mathcal{X}_C^* . If true, this map would give an explicit toric isomorphism between the two different mirror constructions for the family of quartic hypersurfaces in $G(2, 4)$, since in [3] the mirror family is given by small resolution of \mathcal{X}_C^* .

In the last section, we look at another example to show that the same method can give some results in other cases.

2 Notations and conventions

Throughout the paper, we will set $M' = \mathbb{Z}^5$, $M = \mathbb{Z}^4$, $M_{\mathbb{R}} = M \otimes \mathbb{R}$, and $M'_{\mathbb{R}} = M' \otimes \mathbb{R}$. Also define the dual spaces $N = \text{Hom}(M, \mathbb{Z})$, $N' = \text{Hom}(M', \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$, and $N'_{\mathbb{R}} = N' \otimes \mathbb{R}$.

We will denote the convex hull of the sets S_1, \dots, S_n in a real vector space by $\text{Conv}(S_1, \dots, S_n)$. If S is any set in a real vector space, we will use $C(S)$ for the cone over S based at the origin, or

$$C(S) = \{rs \mid s \in S, r \in \mathbb{R}, r \geq 0\}.$$

If P is a compact convex polytope in a real vector space X with the origin in its interior, we will use $\Sigma(P)$ to denote the fan consisting of cones over the proper faces of P . We will also sometimes use $X(P)$ to denote the toric variety associated to $\Sigma(P)$. If Σ is a fan, then $X(\Sigma)$ will denote the toric variety associated to Σ . The dual polytope of P , P^* , is contained in the dual vector space X^* and defined as

$$\{x \in X^* \mid \langle p, x \rangle \geq -1, \forall p \in P\}.$$

Here, $\langle n, m \rangle$ denotes the natural real-valued pairing between X and X^* .

We define the Newton polytope of a real-valued function φ on a real vector space X by

$$Newt(\varphi) = \{n \in X^* \mid \langle n, m \rangle \geq -\varphi(m), \forall m \in X\}.$$

In the other direction, given a compact convex polytope $\Delta \subseteq X^*$, we may define the associated upper convex function $\varphi : X \rightarrow \mathbb{R}$ by

$$\varphi(x) = -\min \{ \langle p, x \rangle \mid p \in \Delta \}.$$

(All convex functions in this paper will be upper convex, meaning that

$$\varphi(ax + by) \leq a\varphi(x) + b\varphi(y)$$

for any $x, y \in X$ and $0 \leq a, b \leq 1, a + b = 1$.) These operations give a one-to-one correspondence between continuous real-valued upper convex piecewise linear functions on X and compact convex polytopes in X^* . Furthermore, under the correspondence, Minkowski sum of polytopes corresponds to simple addition of real valued functions. See, e.g., [8], Theorem A.18.

If φ is a continuous real-valued upper convex piecewise linear function on X , then the maximal domains of linearity of φ will decompose X into a union of polyhedral cones making up a fan which we will call Σ_φ . (These cones may not be strictly convex, in other words, they may contain a nontrivial linear subspace of X .)

3 Review of mirror construction via conifold transitions

The Plücker embedding of $G(2, 4)$ in \mathbb{P}^5 is defined by the single quadratic equation

$$z_0 z_1 - z_2 z_3 + z_4 z_5 = 0$$

where z_0, \dots, z_5 are homogeneous coordinates on \mathbb{P}^5 . Intersecting $G(2, 4)$ with a generic degree four hypersurface in \mathbb{P}^5 gives a smooth Calabi-Yau threefold which is part of the family of $(2, 4)$ complete intersections in \mathbb{P}^5 .

The method used by [3] to construct the mirror of this family of threefolds starts by degenerating $G(2, 4)$ to the toric variety $P(2, 4)$ via the equation

$$tz_0 z_1 - z_2 z_3 + z_4 z_5 = 0$$

in $\mathbb{P}^5 \times \mathbb{C}$. Fixing a value of t with $t \neq 0$ results in a variety isomorphic to $G(2, 4)$. When $t = 0$ we get the singular toric variety $P(2, 4)$, defined by the equation $z_2 z_3 = z_4 z_5$. The singular locus of $P(2, 4)$ consists of the line of nodes $z_2 = z_3 = z_4 = z_5 = 0$, and a generic degree four hypersurface in \mathbb{P}^5 will intersect the line transversely in four points, resulting in a Calabi-Yau variety with four nodes as singularities.

There exist toric resolutions of $P(2, 4)$ to a nonsingular toric variety $\hat{P}(2, 4)$. The toric variety $\hat{P}(2, 4)$ can be obtained by blowing up $P(2, 4)$ on a toric divisor of the form $z_i = z_j = 0$ for any choice of $i \in \{2, 3\}$, $j \in \{4, 5\}$. Such a resolution induces a small resolution of the four nodes on a generic Calabi-Yau hypersurface in $P(2, 4)$. The degeneration of a Calabi-Yau hypersurface X in $G(2, 4)$ to a hypersurface in $P(2, 4)$, followed by this small resolution, is the conifold transition used by [3] to construct the mirror of X .

To perform the mirror conifold transition, [3] starts with the mirror family to the family $\mathcal{X}_{\hat{P}(2, 4)}$ of Calabi-Yau hypersurfaces in $\hat{P}(2, 4)$ as provided by Batyrev's construction in [1]. The fan for $P(2, 4)$ may be described as the cones over faces of a reflexive polytope $\Delta_{P(2, 4)} \subseteq M_{\mathbb{R}} \cong \mathbb{R}^4$ with six vertices, defined as

$$\Delta_{P(2, 4)} = \text{Conv}(u_1, u_2, u_3, u_4, u_5, u_6)$$

where the vertices u_i are defined as

$$u_1 = f_1, \quad u_2 = f_2, \quad u_3 = f_3,$$

$$u_4 = f_4, \quad u_5 = -f_1 - f_2 - f_3, \quad u_6 = f_1 + f_2 - f_4,$$

with f_1, \dots, f_4 being the standard basis of $M_{\mathbb{R}}$. According to Batyrev's construction, the mirror family to $\mathcal{X}_{\hat{P}(2, 4)}$ is given by a Calabi-Yau compactification of the subvariety of the torus $T = \text{Spec } \mathbb{C}[M]$ defined by

$$-1 + a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 (X_1 X_2 X_3)^{-1} + a_6 X_1 X_2 X_4^{-1} = 0$$

where the a_i are generic coefficients and $X_i = z^{f_i}$. The degenerate subfamily of this family is given by the additional equation $a_1 a_2 = a_4 a_6$. This equation arises from the fact that X_1 , X_2 , X_4 and $X_1 X_2 X_4^{-1}$ are the monomials corresponding to the vertices of the square face $\text{Conv}(u_1, u_2, u_4, u_6)$ of $\Delta_{P(2, 4)}$, with $\{u_1, u_2\}$ and $\{u_4, u_6\}$ being pairs of opposite corners. The toric stratum of $P(2, 4)$ corresponding to the cone over this face is the singular

locus of $P(2, 4)$ and the location of the nodes of a generic Calabi-Yau quartic hypersurface in $P(2, 4)$.

By Batyrev's construction, a Calabi-Yau compactification of this family exists in the toric variety given by an MPCP resolution of the toric variety $X(\Delta_{P(2,4)}^*)$, which is defined by the fan of cones over the faces of the dual polytope $\Delta_{P(2,4)}^* \subseteq N_{\mathbb{R}}$. The face of $\Delta_{P(2,4)}^*$ that is dual to the square $\text{Conv}(u_1, u_2, u_4, u_6)$ is the line segment

$$L = \text{Conv}((-1, -1, -1, -1), (-1, -1, 3, -1)),$$

using the basis for $N_{\mathbb{R}}$ which is dual to f_1, \dots, f_4 . After an MPCP resolution of $X(\Delta_{P(2,4)}^*)$, L will be subdivided into four smaller line segments. The open subset of the MPCP resolution corresponding to the union of the cones over these line segments will contain four nodes of a generic Calabi-Yau hypersurface in the degenerate subfamily. A small resolution of these nodes yields the mirror of the original Calabi-Yau manifold X .

As in [3], using the $(\mathbb{C}^*)^4$ action on T we may assume without loss of generality that $a_1 = a_2 = a_3 = a_4 = 1$, so that the family becomes

$$-1 + X_1 + X_2 + X_3 + X_4 + a_5(X_1 X_2 X_3)^{-1} + a_6 X_1 X_2 X_4^{-1} = 0 \quad (1)$$

and the degenerate subfamily is defined by $a_6 = 1$. We will refer to this degenerate family as \mathcal{X}_C^* .

4 The Batyrev-Borisov mirror and its birational equivalence

Now we establish a birational morphism between the family \mathcal{X}_C^* and the mirror family \mathcal{X}_{BB}^* to $(2, 4)$ hypersurfaces in \mathbb{P}^5 as given by the Batyrev-Borisov construction.

Let e_1, \dots, e_5 be the standard basis for $M'_{\mathbb{R}} \cong \mathbb{R}^5$ and let $\Delta \subseteq M'_{\mathbb{R}}$ be the reflexive polytope $\text{Conv}(e_1, \dots, e_5, -e_1 - e_2 - e_3 - e_4 - e_5)$. Then the fan $\Sigma(\Delta)$ consisting of cones over the proper faces of Δ is the fan for \mathbb{P}^5 . Define the piecewise linear functions φ_1, φ_2 on $\Sigma(\Delta)$ by $\varphi_1(e_i) = 1$ for $i = 1, 2, 3, 4$, and $\varphi_1(e_5) = \varphi_1(-e_1 - \dots - e_5) = 0$, and $\varphi_2(e_i) = 0$ for $i = 1, 2, 3, 4$, and $\varphi_2(e_5) = \varphi_2(-e_1 - \dots - e_5) = 1$. The line bundles on \mathbb{P}^5 associated to φ_1 and φ_2 are $\mathcal{O}_{\mathbb{P}^5}(4)$ and $\mathcal{O}_{\mathbb{P}^5}(2)$ respectively.

The Batyrev-Borisov construction now states that the mirror family of $(2, 4)$ complete intersections in \mathbb{P}^5 is a Calabi-Yau compactification of the subvariety of $T' = \text{Spec } \mathbb{C}[M']$ defined by

$$\begin{aligned} b_1 Y_1 + b_2 Y_2 + b_3 Y_3 + b_4 Y_4 &= 1 \\ b_5 Y_5 + b_6 (Y_1 Y_2 Y_3 Y_4 Y_5)^{-1} &= 1 \end{aligned}$$

with the b_i generic coefficients and $Y_i = z^{e_i}$. By the $(\mathbb{C}^*)^5$ action on T' we may assume without loss that $b_1 = b_2 = b_3 = b_5 = b_6 = 1$, so the equations reduce to

$$\begin{aligned} Y_1 + Y_2 + Y_3 + b_4 Y_4 &= 1 \\ Y_5 + (Y_1 Y_2 Y_3 Y_4 Y_5)^{-1} &= 1. \end{aligned}$$

We will refer to the second family as \mathcal{X}_{BB}^* .

In the following theorem, we will define a birational equivalence between the families \mathcal{X}_{BB}^* and \mathcal{X}_C^* by defining a map between the tori T' and T . These families should properly be considered to lie in $T' \times \mathbb{C}$ and $T \times \mathbb{C}$, with the \mathbb{C} factor representing the parameters b_4 and a_5 , respectively, and a morphism between the families could be given as a map between $T' \times \mathbb{C}$ and $T \times \mathbb{C}$ which maps \mathcal{X}_{BB}^* to \mathcal{X}_C^* and respects projection to \mathbb{C} . However, in this case, it is possible to realize the map as a single map from T' to T ; in other words, the map does not depend on the parameter b_4 .

Theorem 4.1. *After setting $b_4 = a_5$, the families \mathcal{X}_{BB}^* and \mathcal{X}_C^* are birationally equivalent via the map $f : T' \rightarrow T$ defined by $X_1 = (Y_2 Y_3 Y_4 Y_5)^{-1}$, $X_2 = Y_2 Y_5$, $X_3 = Y_3$, $X_4 = (Y_1 Y_3 Y_4 Y_5)^{-1}$.*

Proof. First we observe that the defining equation for \mathcal{X}_C^* , equation (1), may be factored as

$$-1 + (1 + X_4 X_2^{-1})(X_4^{-1} X_1 X_2 + X_2) + X_3 + a_5 (X_1 X_2 X_3)^{-1} = 0.$$

After doing this, it is straightforward to check that the rational map from T to T' defined by

$$\begin{aligned} Y_1 &= (1 + X_4 X_2^{-1})(X_4^{-1} X_1 X_2) \\ Y_2 &= (1 + X_4 X_2^{-1}) X_2 \\ Y_3 &= X_3 \\ Y_4 &= (X_1 X_2 X_3)^{-1} \\ Y_5 &= (1 + X_4 X_2^{-1})^{-1} \end{aligned}$$

when restricted to any member of \mathcal{X}_C^* , is a birational inverse for the restriction of f to the corresponding member of \mathcal{X}_{BB}^* . \square

5 Extension to the ambient toric varieties

Because the map $f : T' \rightarrow T$ is defined by monomials, it is associated to a \mathbb{Z} -linear map $g : M \rightarrow M'$ and its dual map $g^* : N' \rightarrow N$. In the standard bases of M and M' , g is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 \end{pmatrix}$$

acting on the left (with elements of M and M' written as column vectors), and in the dual bases of N and N' , g^* is defined by its transpose. The kernel of g^* is easily seen to be the \mathbb{Z} -span of the vector $(1, 1, 0, 0, -1)$. We will also use g and g^* to denote the extensions of these \mathbb{Z} -linear maps to maps from $M_{\mathbb{R}}$ to $M'_{\mathbb{R}}$ and $N'_{\mathbb{R}}$ to $N_{\mathbb{R}}$.

Remark 5.1. Although our focus for the rest of the paper will be g^* acting as a map between the toric varieties on the mirror side, one can check that the map $g : M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}}$ defines a map of fans $g : \Sigma(\Delta_{P(2,4)}) \rightarrow \Sigma(\Delta)$ which embeds $P(2, 4)$ into \mathbb{P}^5 as the zero locus of the equation

$$z^{e_1^* + e_2^* - e_5^*} = z^0.$$

Here e_1^*, \dots, e_5^* is the dual basis to the standard basis e_1, \dots, e_5 of $M'_{\mathbb{R}}$. Because $e_1^* + e_2^* - e_5^*$ and 0 are both contained in $Newt(\varphi_2)$, we may consider both sides of the equation to lie in $\mathcal{O}_{\mathbb{P}^5}(2)$, with $z^{e_1^* + e_2^* - e_5^*}$ the quadratic monomial vanishing on the divisors of \mathbb{P}^5 corresponding to the vertices e_1 and e_2 of Δ , and z^0 the monomial vanishing on the divisors corresponding to e_5 and $-e_1 - e_2 - e_3 - e_4 - e_5$. (Also note that $e_1^* + e_2^* - e_5^*$ is the kernel of g^* .)

Once the family \mathcal{X}_{BB}^* is compactified and resolved by the Batyrev-Borisov construction, it will lie in an MPCP resolution of a certain Gorenstein toric Fano variety with a fan consisting of cones over a polytope $\nabla \subseteq N'_{\mathbb{R}}$. With $\varphi_1, \varphi_2 : M'_{\mathbb{R}} \rightarrow \mathbb{R}$ defined as before, set $\nabla_1 = Newt(\varphi_1)$ and $\nabla_2 = Newt(\varphi_2)$. Then the Batyrev-Borisov construction in [2] says that $\nabla = Conv(\nabla_1, \nabla_2)$.

Later we will also make use of the polytopes Δ_i for $i = 1, 2$, which the Batyrev-Borisov construction defines as the convex hull of the origin and the vertices of Δ where $\varphi_i = 1$.

One may verify that in the basis of $N'_{\mathbb{R}}$ that is dual to the standard basis of $M'_{\mathbb{R}} \cong \mathbb{R}^5$, the vertices of ∇_1 are the rows of the matrix

$$\begin{pmatrix} -1 & -1 & -1 & -1 & 0 \\ 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$$

and the vertices of ∇_2 are the rows of

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The vertices of ∇ consist of the twelve rows of the above two matrices.

We will need the following convex geometry result:

Proposition 5.2. *Let $p : X \rightarrow Y$ be a linear map between finite dimensional real vector spaces, and let φ be a real-valued upper convex function on Y . Let $p^* : Y^* \rightarrow X^*$ be the dual map. Then $\text{Newt}(\varphi \circ p) = p^*(\text{Newt}(\varphi))$.*

Proof. First we show the containment $p^*(\text{Newt}(\varphi)) \subseteq \text{Newt}(\varphi \circ p)$. We must show that for any $y \in \text{Newt}(\varphi)$, that $\langle p^*(y), z \rangle \geq -\varphi(p(z))$ for all $z \in X$. This is true because $\langle p^*(y), z \rangle = \langle y, p(z) \rangle$, and $\langle y, p(z) \rangle \geq -\varphi(p(z))$ because $y \in \text{Newt}(\varphi)$.

To show the reverse containment, let $x \in \text{Newt}(\varphi \circ p)$. We first claim that because x is such that $\langle x, z \rangle \geq -\varphi(p(z))$ for all $z \in X$, then $x \in p^*(Y^*)$. The latter condition is equivalent to x , regarded as a function on X , being constant on the fibers of p , which follows from the first condition because $-\varphi(p(z))$ is constant on the fibers of p and x is linear. Therefore any x satisfying the first condition descends to a function x' on $p(X) \subseteq Y$, and is completely determined by its values on this subspace. We choose a

complementary subspace $S \subseteq Y$ to $p(M)$ and choose a linear function L on S such that $L(y) \geq -\varphi(y)$ for all $y \in S$, which is possible since φ is upper convex. Let L' be the unique linear function such that $L'|_S = L$ and $L'|_{p(X)} = x'$. Then by convexity of φ , $L'(y) \geq -\varphi(y)$ for all $y \in Y$. Regarding L' as an element of Y^* , this means that $L' \in \text{Newt}(\varphi)$, and we have that $p^*(L') = x$ since $L'|_{p(X)} = x'$, which proves the containment. \square

Lemma 5.3. *With $g^* : N'_\mathbb{R} \rightarrow N_\mathbb{R}$ defined as above, $g^*(\nabla) = \Delta_{P(2,4)}^*$.*

Proof. First we apply Proposition 5.2 with $\varphi = \varphi_1$ and $p = g$. It is straightforward to check that $\varphi_1 \circ g$ is the piecewise linear function h_1 on $\Sigma(\Delta_{P(2,4)})$ associated to the anticanonical bundle of $P(2,4)$, in other words, the piecewise linear function which is equal to 1 on the primitive integral generators of all rays in the fan, u_i for $1 \leq i \leq 6$. Since $\Delta_{P(2,4)}^* = \text{Newt}(h_1)$, the proposition gives us that $g^*(\text{Newt}(\varphi_1)) = \Delta_{P(2,4)}^*$.

One can also check that $\varphi_2 \circ g$ gives a piecewise linear function h_2 on $\Sigma(\Delta_{P(2,4)})$ which is equal to one on u_1, u_2, u_4 and u_6 (the vertices of the square face of $\Delta_{P(2,4)}$), and zero on all other rays. Because $h_2 \leq h_1$,

$$g^*(\text{Newt}(\varphi_2)) = \text{Newt}(h_2) \subseteq \text{Newt}(h_1) = \Delta_{P(2,4)}^*.$$

Thus we have that

$$\begin{aligned} g^*(\nabla) &= \text{Conv}(g^*(\text{Newt}(\varphi_1)), g^*(\text{Newt}(\varphi_2))) = \\ &= \text{Conv}(g^*(\text{Newt}(\varphi_1))) = \Delta_{P(2,4)}^*. \end{aligned}$$

\square

In spite of the above result, g^* does not determine a map between the toric varieties $X(\nabla)$ and $X(\Delta_{P(2,4)}^*)$ associated to $\Sigma(\nabla)$ and $\Sigma(\Delta_{P(2,4)}^*)$, because the images under g^* of some cones in $\Sigma(\nabla)$ are not contained in cones of $\Sigma(\Delta_{P(2,4)}^*)$. However, we will show that g^* does induce a map from an open toric subvariety of an MPCP resolution of $X(\nabla)$ to an MPCP resolution of $X(\Delta_{P(2,4)}^*)$. Because this open toric subvariety will contain the generic member of the family $\mathcal{X}_{BB}^* \subseteq X(\nabla)$, this will be sufficient for our purposes.

This open toric subvariety of $X(\nabla)$ will consist of a resolution of a fan Σ' which is a subfan of $\Sigma(\nabla)$, and whose maximal cones are a subset of the three-dimensional cones of $\Sigma(\nabla)$. Since the three-dimensional cones of $\Sigma(\nabla)$ are the cones over two-dimensional faces of ∇ , first we characterize the two-dimensional faces of ∇ . This can be done by checking a computer generated

list of the 50 two-faces of ∇ . A non-computer-aided proof is also relatively easy but we omit it as it is not very relevant to the rest of the paper.

Proposition 5.4. *If $f \subseteq \nabla$ is a two-dimensional face, then exactly one of the following holds:*

1. *f is a two-dimensional face of ∇_1 or ∇_2 .*
2. *f is of the form $\text{Conv}(g_1, g_2)$ where g_1 is a one-dimensional face of ∇_1 and g_2 is a one-dimensional face of ∇_2 , and g_1 and g_2 are parallel.*

Note that these are not sufficient conditions for a set f to be a two-face of ∇ , as there are 55 such sets possible but only 50 two-faces of ∇ .

In preparation for describing the fan Σ' , we next need to eliminate certain three dimensional cones of $\Sigma(\nabla)$, which correspond to torus orbits of $\Sigma(\nabla)$ not intersecting generic members of the family \mathcal{X}_{BB}^* . An easier version of the same argument can be used to verify that no codimension zero or one toric strata of $X(\nabla)$ intersect generic members of \mathcal{X}_{BB}^* , as would be expected since the dimension of the toric variety is five and the dimension of the Calabi-Yau manifolds is three. Thus, all four and five dimensional cones may be removed from $\Sigma(\nabla)$.

Recall that if $\mathcal{L}(P)$ is a line bundle on a toric variety X associated to the polytope P (so that P is the Newton polytope of global sections of $\mathcal{L}(P)$), and $m \in P$ is a lattice point, then m represents a monomial global section z^m of $\mathcal{L}(P)$ and there is a piecewise linear function $\varphi_m \geq 0$ on the fan for X whose Newton polytope $\text{Newt}(\varphi_m)$ is $P - m$. The function φ_m describes the zero locus of z^m on X , which is the union of toric divisors of X consisting of divisors associated to rays on which φ_m is nonzero. The section z^m is identically zero on a toric stratum of X if and only if φ_m is not identically zero on the cone C in the fan for X associated to the toric stratum. Conversely, if φ_m is identically zero on the cone, then it is nowhere vanishing on the torus orbit T_C which is dense in the stratum.

Lemma 5.5. *Suppose $T_{C(f)}$ is a torus orbit of $X(\nabla)$ associated to the cone $C(f)$ in the fan $\Sigma(\nabla)$, where f is a face of ∇ . If, for either $i = 1$ or $i = 2$, the functions $\varphi_{m_1}, \dots, \varphi_{m_k}$ associated to the monomial global sections of $\mathcal{L}(\Delta_i)$, where m_1, \dots, m_k is a list of all lattice points in Δ_i , are such that a single φ_{m_j} is identically zero on $C(f)$ but all others are not, then a generic member of \mathcal{X}_{BB}^* does not intersect the torus orbit $T_{C(f)}$.*

Proof. The members of \mathcal{X}_{BB}^* are defined by the equations $s_1 = s_2 = 0$ where s_i is a global section of $\mathcal{L}(\Delta_i)$. Each section s_i may be represented as a sum

$$\sum_{m \in (\Delta_i \cap N')} c_m z^m$$

with $c_m \in \mathbb{C}$. By the hypothesis of the proposition statement, at any point in the torus orbit $T_{C(f)}$, for either $i = 1$ or 2 , all but one of the monomial global sections of $\mathcal{L}(\Delta_i)$ will vanish. If the nonvanishing section is z^{m_j} , then the equation $s_i = 0$ collapses to $c_{m_j} z^{m_j} = 0$. For generic members of \mathcal{X}_{BB}^* , $c_{m_j} \neq 0$, and since z^{m_j} does not vanish at any point of $T_{C(f)}$, $s_i = 0$ has no solutions on $T_{C(f)}$ and we obtain the result. \square

Now we describe which three-dimensional cones $C(f) \in \Sigma(\nabla)$ satisfy the above conditions. If a cone C in a fan is not contained by any other cones, then we may remove the cone from the fan to obtain a new fan and toric variety, which is simply the old toric variety with the torus orbit associated to C removed. Since we will have already removed all four and five dimensional cones of $\Sigma(\nabla)$, this means that a three-dimensional cone satisfying the conditions of Lemma 5.5 may also be removed from $\Sigma(\nabla)$.

Proposition 5.6. *Suppose that f is a 2-face of ∇ satisfying either of the following conditions:*

1. f contains $(0, 0, 0, 0, 1)$ or $(0, 0, 0, 0, -1)$.
2. f contains any three of the rows of

$$\begin{pmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & -1 & -1 & 3 & 0 \end{pmatrix}$$

Then the cone $C(f)$ satisfies the conditions of Lemma 5.5 and the associated torus orbit does not contain any points of generic members of \mathcal{X}_{BB}^ .*

Proof. This is straightforward to deduce from the following descriptions of piecewise linear functions giving vanishing data for sections of $\mathcal{L}(\Delta_i)$, $i = 1, 2$. Here φ_m^i denotes the piecewise linear function on $\Sigma(\nabla)$ with Newton polytope $\Delta_i - m$. These may be easily calculated from the fact that φ_0^i equals 1 on all vertices of ∇_i and equals 0 on all other vertices of ∇ . Each

of the piecewise linear functions below is nonzero on exactly two of the rays in $\Sigma(\nabla)$ and we list the nonzero values only. We have:

$$\begin{aligned}
\varphi_{e_5}^1(-1, -1, -1, -1, 4) &= 4 \\
\varphi_{e_5}^1(0, 0, 0, 0, 1) &= 2 \\
\varphi_{-e_1-e_2-e_3-e_4-e_5}^1(-1, -1, -1, -1, 0) &= 4 \\
\varphi_{-e_1-e_2-e_3-e_4-e_5}^1(0, 0, 0, 0, -1) &= 2 \\
\varphi_{e_1}^2(3, -1, -1, -1, 0) &= 4 \\
\varphi_{e_1}^2(2, 0, 0, 0, -1) &= 2 \\
\varphi_{e_2}^2(-1, 3, -1, -1, 0) &= 4 \\
\varphi_{e_2}^2(0, 2, 0, 0, -1) &= 2 \\
\varphi_{e_3}^2(-1, -1, 3, -1, 0) &= 4 \\
\varphi_{e_3}^2(0, 0, 2, 0, -1) &= 2 \\
\varphi_{e_4}^2(-1, -1, -1, 3, 0) &= 4 \\
\varphi_{e_4}^2(0, 0, 0, 2, -1) &= 2
\end{aligned}$$

□

We define the fan Σ' as the subfan of $\Sigma(\nabla)$ with all four and five dimensional cones removed, as well as the three dimensional cones $C(f)$ where f is a 2-face of ∇ satisfying the conditions of Proposition 5.6. Also define $\Sigma^{[3]}(\Delta_{P(2,4)}^*)$, as in [1], as the subfan of $\Sigma(\Delta_{P(2,4)}^*)$ consisting of cones of dimension ≤ 3 . In the following proposition, a crepant subdivision of Σ' will mean a subdivision of Σ' such that each cone is the cone over a polytope $\text{Conv}(v_1, \dots, v_k)$ with $v_1, \dots, v_k \in M'$ contained in some common proper face of ∇ .

Proposition 5.7. *There exists a crepant subdivision Σ'' of the fan Σ' such that $g^* : \Sigma'' \rightarrow \Sigma^{[3]}(\Delta_{P(2,4)}^*)$ exists, that is, for every cone $C \in \Sigma''$, $g^*(C)$ is contained in a cone of $\Sigma^{[3]}(\Delta_{P(2,4)}^*)$.*

Proof. We will approach the proof by examining the images $g^*(f)$ of the two dimensional faces $f \subseteq \nabla$ such that $C(f) \in \Sigma'$, which are described by Propositions 5.6 and 5.4. First we deal with the two-faces of ∇_1 . Under g^* , by the proof of Lemma 5.3, the image of ∇_1 is $\Delta_{P(2,4)}^*$. Now, the face configuration of the polytope $\Delta_{P(2,4)}^*$ is combinatorially the same as $\Delta_{P(2,4)}$

(although it contains more lattice points) so it has six vertices. Since ∇_1 also has six vertices, they must be mapped bijectively to the vertices of $\Delta_{P(2,4)}^*$. The two-faces of ∇_1 consist of the convex hulls of any three of its vertices, totaling $\binom{6}{3} = 20$ faces.

The vertices of $\Delta_{P(2,4)}^*$ are the rows of the matrix

$$\begin{pmatrix} -1 & -1 & -1 & -1 \\ 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 3 & -1 & -1 & 3 \\ -1 & 3 & -1 & 3 \end{pmatrix}$$

We denote the i -th row of the matrix by v_i . There are thirteen two-faces of $\Delta_{P(2,4)}^*$, consisting of the following:

1. The square $S_1 = \text{Conv}(v_2, v_3, v_5, v_6)$.
2. The eight triangles T_1, \dots, T_8 formed by the convex hull of one of $\{v_1, v_4\}$, one of $\{v_2, v_6\}$ and one of $\{v_3, v_5\}$.
3. The four triangles T_9, \dots, T_{12} formed by the convex hull v_1, v_4 , and one of $\{v_2, v_3, v_5, v_6\}$.

Suppose that the four two-faces of ∇_1 of the form in part 2 of Proposition 5.6 are removed. Then one may check directly that of the remaining sixteen two-faces of ∇_1 , there are twelve two-faces U_1, \dots, U_{12} such that g^* maps U_i to T_i bijectively. The remaining four two-faces are mapped to the triangles contained in S_1 given by taking the convex hull of any three of the vertices $\{v_2, v_3, v_5, v_6\}$. Thus we see that the images of the cones $C(f) \in \Sigma'$ for f a two-face of ∇_1 are already contained in cones of $\Sigma^{[3]}(\Delta_{P(2,4)}^*)$ and no subdivision of these cones is needed.

Now we turn to the remaining two-faces of ∇ , which by Proposition 5.4 are either two-faces of ∇_2 or contain a one-face of ∇_2 . The two vertices of ∇_2 , $(0, 0, 0, 0, \pm 1)$, as well as all faces of ∇_2 containing them, may be removed by Proposition 5.6. Thus we are left with four vertices of ∇_2 , and we denote by ∇'_2 their convex hull. We must examine the two-faces of ∇'_2 , and the six two-faces of ∇ consisting of convex hulls of the six one-faces of ∇'_2 and their parallel one-faces in ∇_1 .

First we deal with the two-faces of ∇'_2 . The vertices of ∇'_2 are the rows

of the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \end{pmatrix}$$

with image under g^* equal to

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & 0 & -1 \end{pmatrix}.$$

We label the rows of the second matrix t_1, \dots, t_4 . In terms of the faces of $\Delta_{P(2,4)}^*$, t_1 is contained in the relative interior of $\text{Conv}(v_1, v_2, v_4)$, t_2 is contained in the relative interior of $\text{Conv}(v_1, v_4, v_6)$, and t_3 and t_4 are contained in the line segment from v_1 to v_4 . Thus, $\text{Conv}(t_1, t_3, t_4)$ and $\text{Conv}(t_2, t_3, t_4)$ are already contained in the faces $\text{Conv}(v_1, v_2, v_4)$ and $\text{Conv}(v_1, v_4, v_6)$, respectively, of $\Delta_{P(2,4)}^*$, and no subdivision of the corresponding faces of ∇'_2 is needed. On the other hand, $\text{Conv}(t_1, t_2, t_3)$ and $\text{Conv}(t_1, t_2, t_4)$ contain the origin, since $t_1 = -t_2$. The faces of ∇'_2 mapping to these convex hulls are

$$e_1 = \text{Conv}((2, 0, 0, 0, -1), (0, 2, 0, 0, -1), (0, 0, 2, 0, -1))$$

and

$$e_2 = \text{Conv}((2, 0, 0, 0, -1), (0, 2, 0, 0, -1), (0, 0, 0, 2, -1)).$$

If these faces are each cut in half by adding $(1, 1, 0, 0, -1)$ as a vertex, which is in the kernel of g^* , and

$$\text{Conv}((1, 1, 0, 0, -1), (0, 0, 2, 0, -1))$$

or

$$\text{Conv}((1, 1, 0, 0, -1), (0, 0, 0, 2, -1))$$

as line segments (see Figure 1), then the faces

$$f_1 = \text{Conv}((2, 0, 0, 0, -1), (1, 1, 0, 0, -1), (0, 0, 2, 0, -1))$$

$$f_2 = \text{Conv}((2, 0, 0, 0, -1), (1, 1, 0, 0, -1), (0, 0, 0, 2, -1))$$

are mapped to $\text{Conv}(0, t_1, t_3)$ and $\text{Conv}(0, t_1, t_4)$ respectively, which are the

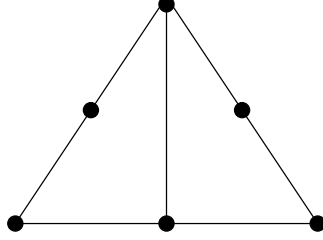


Figure 1: Subdivision used for the faces e_1 and e_2 .

convex hull of the origin with line segments contained in $\text{Conv}(v_1, v_2, v_4)$. This implies that the cones $C(f_1)$ and $C(f_2)$ are mapped to cones contained in $\Sigma^{[3]}(\Delta_{P(2,4)}^*)$. In a similar fashion, the cones over the other two faces

$$\begin{aligned} &\text{Conv}((0, 2, 0, 0, -1), (1, 1, 0, 0, -1), (0, 0, 2, 0, -1)) \\ &\text{Conv}((0, 2, 0, 0, -1), (1, 1, 0, 0, -1), (0, 0, 0, 2, -1)) \end{aligned}$$

will be mapped to cones contained in $\Sigma^{[3]}(\Delta_{P(2,4)}^*)$.

Finally we must deal with the six faces which are convex hulls of the one-faces of ∇'_2 and the parallel one-faces in ∇_1 . It is easy to check that the faces

$$\begin{aligned} &\text{Conv}((2, 0, 0, 0, -1), (0, 0, 2, 0, -1), (3, -1, -1, -1, 0), (-1, -1, 3, -1, 0)) \\ &\text{Conv}((2, 0, 0, 0, -1), (0, 0, 0, 2, -1), (3, -1, -1, -1, 0), (-1, -1, -1, 3, 0)) \end{aligned}$$

are mapped to subsets of $\text{Conv}(v_1, v_2, v_4)$, and the faces

$$\begin{aligned} &\text{Conv}((0, 2, 0, 0, -1), (0, 0, 2, 0, -1), (-1, 3, -1, -1, 0), (-1, -1, 3, -1, 0)) \\ &\text{Conv}((0, 2, 0, 0, -1), (0, 0, 0, 2, -1), (-1, 3, -1, -1, 0), (-1, -1, -1, 3, 0)) \end{aligned}$$

are mapped to subsets of $\text{Conv}(v_1, v_4, v_6)$. Thus, neither of these faces needs to be subdivided. The face

$$e_3 = \text{Conv}((0, 0, 2, 0, -1), (0, 0, 0, 2, -1), (-1, -1, 3, -1, 0), (-1, -1, -1, 3, 0))$$

is mapped to the line segment $\text{Conv}(v_1, v_4)$, and also does not need to be subdivided. The remaining face

$$e_4 = \text{Conv}((2, 0, 0, 0, -1), (0, 2, 0, 0, -1), (3, -1, -1, -1, 0), (-1, 3, -1, -1, 0))$$

contains $(1, 1, 0, 0, -1)$ which is mapped to the origin. If we subdivide this face into three faces by adding the vertex $(1, 1, 0, 0, -1)$ and the two line segments

$$\begin{aligned} & \text{Conv}((1, 1, 0, 0, -1), (3, -1, -1, -1, 0)) \\ & \text{Conv}((1, 1, 0, 0, -1), (-1, 3, -1, -1, 0)), \end{aligned}$$

(see Figure 2) then

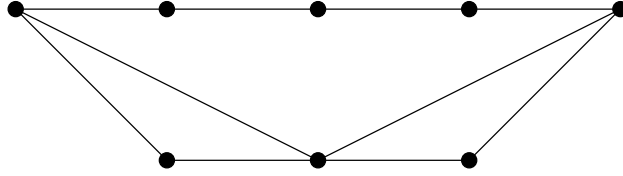


Figure 2: Subdivision used for the face e_4 .

$$\text{Conv}((1, 1, 0, 0, -1), (2, 0, 0, 0, -1), (3, -1, -1, -1, 0))$$

is mapped to the convex hull of the origin with a line segment in

$$\text{Conv}(v_1, v_2, v_4),$$

and

$$\text{Conv}((1, 1, 0, 0, -1), (0, 2, 0, 0, -1), (-1, 3, -1, -1, 0))$$

is mapped to the convex hull of the origin with a line segment in

$$\text{Conv}(v_1, v_4, v_6).$$

Finally,

$$\text{Conv}((3, -1, -1, -1, 0), (-1, 3, -1, -1, 0), (1, 1, 0, 0, -1))$$

is mapped to the convex hull of the origin and the line segment $\text{Conv}(v_2, v_6)$ which is a diagonal of the square S_1 . \square

This is not quite enough to prove that g^* induces a map between the (partially) resolved and compactified Calabi-Yaus in the families \mathcal{X}_{BB}^* and \mathcal{X}_C^* , because we must show that the map of toric varieties induced by g^* :

$\Sigma'' \rightarrow \Sigma^{[3]}(\Delta_{P(2,4)}^*)$ lifts to a map between MPCP resolutions of the toric varieties. (This is a slight abuse of terminology since Σ'' and $\Sigma^{[3]}(\Delta_{P(2,4)}^*)$ are not fans given by the cones over faces of a reflexive polytope. By an MPCP resolution of these toric varieties we will mean a subdivision of the fan Σ'' such that each cone in the subdivision is a cone in the fan for an MPCP resolution of $X(\nabla)$, and likewise a subdivision of $\Sigma^{[3]}(\Delta_{P(2,4)}^*)$ such that each cone in the subdivision is a cone in the fan for an MPCP resolution of $X(\Delta_{P(2,4)}^*)$.)

For this approach to work, we will not be able to arbitrarily choose an MPCP resolution of $X(\Delta_{P(2,4)}^*)$. We will need to choose an MPCP resolution such that the image of each cone in Σ'' under g^* is a union of cones in the fan for the MPCP resolution of $X(\Delta_{P(2,4)}^*)$. This will guarantee that the resolution can be pulled back to an MPCP resolution of $X(\nabla)$ such that the map g^* lifts to a map between the induced resolutions of $X(\Sigma'')$ and $X(\Sigma^{[3]}(\Delta_{P(2,4)}^*))$. We will not attempt to answer the apparently more difficult question of whether g^* lifts to a map between the full MPCP resolutions of $X(\nabla)$ and $X(\Delta_{P(2,4)}^*)$, because this is not needed for a map between the Calabi-Yaus to exist.

To construct the needed MPCP resolution we will use the method of [4]. Let us briefly review the method and how it is used to construct MPCP resolutions of Gorenstein toric Fano varieties. Let $Q = \mathbb{Z}^n$ for some n , and let $Q_{\mathbb{R}} = Q \otimes \mathbb{R}$. Suppose that we have a reflexive polytope $\Delta \subseteq Q_{\mathbb{R}}$, and an upper convex piecewise linear function φ on a crepant subdivision of the fan $\Sigma(\Delta)$. Then the convex hull of the set

$$S = \bigcup_{\ell \in \Delta \cap Q} \{(\ell, r) \mid r \geq \varphi(\ell)\}$$

is equal to the set

$$\{(m, r) \mid m \in \Delta, r \geq \varphi(g)\}.$$

This set is a noncompact polytope extending infinitely upwards, with two types of faces, the “vertical” (unbounded) faces and the “lower” (bounded) faces. (See [5], pp. 20-22, where this type of polytope and its dual fan are analyzed further.) If ℓ is a lattice point of the reflexive polytope, and ϵ is a small rational number, consider the set $S' = \text{Conv}(S, (\ell, \varphi(\ell) - \epsilon))$. Provided ϵ is sufficiently small, it can be shown that after projection to $M_{\mathbb{R}}$, the images of all lower faces of S' are contained in images of lower faces of S . Faces of $\text{Conv}(S)$ not containing ℓ remain intact, while (projected) faces f containing

ℓ are subdivided into the faces $\text{Conv}(\ell, f_i)$ as f_i ranges over all codimension-one faces of f not containing ℓ .

In general, if φ is only known to be a piecewise linear function, there is no reason for the lower faces of S' to be the graph of a new piecewise linear function over Δ . However, if φ is known to be strictly convex on a subdivision of $\Sigma(\Delta)$, then this will hold. The reason is that the function defined by lower faces of S' will be piecewise linear if and only if every maximal (n -dimensional) face contains the origin. If an n -dimensional face of S' is contained in a cone of $\Sigma(\Delta)$, then it must contain the origin, because otherwise all its projected lattice points would be contained in a proper face of Δ and it could not be n -dimensional.

To construct an MPCP function, we thus start with any function φ which is strictly convex piecewise linear on a partial crepant subdivision of $\Sigma(\Delta)$; for instance, the function φ_Δ which is identically 1 on the boundary of Δ may be used. We then subtract small rational values at any lattice point $\ell \in \Delta$ which is not contained in a ray of the fan on which φ is strictly convex, obtaining a new piecewise linear function φ' . Repeating the procedure, eventually all lattice points of Δ will be contained in some such ray. We scale the piecewise linear function by some sufficiently large integer to guarantee integrality of all its linear parts, and obtain an MPCP function.

We have the following general lemma:

Lemma 5.8. *Let $Q = \mathbb{Z}^n$ for some n , and let $Q_{\mathbb{R}} = Q \otimes \mathbb{R}$. Let φ be a piecewise linear upper convex function on $Q_{\mathbb{R}}$, and let $\Delta \subseteq Q_{\mathbb{R}}$ be a reflexive polytope. Consider the set of vertical rays in $Q_{\mathbb{R}} \oplus \mathbb{R}$,*

$$S = \bigcup_{\ell \in \Delta \cap Q} \{(\ell, r) \mid r \geq \varphi(\ell)\}.$$

Then

1. *The union of lower faces of $\text{Conv}(S)$ is the graph of a piecewise linear upper convex function φ' on $Q_{\mathbb{R}}$ restricted to Δ , and for any $\ell \in \Delta \cap Q$, $\varphi'(\ell) = \varphi(\ell)$.*
2. *φ' is the unique upper convex function which is piecewise linear on a lattice subdivision of Δ (meaning its maximal domains of linearity, after intersection with Δ , are convex hulls of lattice points in Δ) and for which $\varphi'(\ell) = \varphi(\ell)$ for any $\ell \in \Delta \cap Q$.*
3. *If φ_Δ is the piecewise linear function on $Q_{\mathbb{R}}$ which is equal to 1 on the boundary of Δ , then $\varphi' + \varphi_\Delta$ is strictly convex piecewise linear on a crepant*

subdivision of the fan $\Sigma(\Delta)$, where crepant means that each cone in the fan is a cone over $\text{Conv}(v_1, \dots, v_k)$, with v_1, \dots, v_k lattice points contained in a common proper face of Δ .

Proof. We begin by defining φ' as the function on Δ whose graph is given by the lower faces of $\text{Conv}(S)$. First we establish the statement that $\varphi'(\ell) = \varphi(\ell)$ for any $\ell \in \Delta \cap Q$. It is clear that $\varphi'(\ell) \leq \varphi(\ell)$ for any such ℓ , by the definition of $\varphi'(\ell)$ in terms of the convex hull of S . But given any such ℓ , we may also choose a linear function $L : Q_{\mathbb{R}} \rightarrow \mathbb{R}$ such that $L(\ell) = \varphi(\ell)$ and $L \leq \varphi$ on all $Q_{\mathbb{R}}$; for instance, choose the linear function equal to φ on some maximal domain of linearity of φ containing ℓ . Then the half-space

$$\{(q, r) \mid r \geq L(q)\} \subseteq Q_{\mathbb{R}} \oplus \mathbb{R}$$

will contain S , and this implies that $\varphi'(\ell) \geq \varphi(\ell)$. So $\varphi'(\ell) = \varphi(\ell)$.

Now suppose, by way of contradiction, that φ' is not the restriction of a piecewise linear function to Δ . This means that if we set U equal to the union of all maximal (n -dimensional) lower faces of S containing the origin, then the projection of U to $Q_{\mathbb{R}}$, $\pi(U)$, is not equal to all of Δ . Because Δ is a lattice polytope, and $\pi(U)$ is a union of lattice polytopes contained in Δ , there must be some lattice point of Δ missing from $\pi(U)$. If ℓ is such a lattice point, then for small $\epsilon > 0$, $\epsilon\ell$ will be contained in $\pi(U)$. We must have that $\varphi'(\ell) \neq (1/\epsilon)\varphi'(\epsilon\ell)$, otherwise ℓ would be contained in $\pi(U)$, and by convexity we get that $\varphi'(\ell) > (1/\epsilon)\varphi'(\epsilon\ell)$. It is clear from the definition of φ' that $\varphi' \geq \varphi$ on all Δ . Also, because φ is piecewise linear, $(1/\epsilon)\varphi(\epsilon\ell) = \varphi(\ell)$. So we have

$$\varphi'(\ell) > (1/\epsilon)\varphi'(\epsilon\ell) \geq (1/\epsilon)\varphi(\epsilon\ell) = \varphi(\ell)$$

but $\varphi(\ell) = \varphi'(\ell)$ as demonstrated in the first paragraph, so we get that $\varphi'(\ell) > \varphi'(\ell)$, a contradiction.

For the second statement, suppose that f is any upper convex function which is piecewise linear on a lattice subdivision of Δ and equal to φ on all lattice points of Δ . Then it follows from convexity that at any point $p \in \Delta$, $f(p)$ will be equal to the minimum possible value of $c_1\varphi(\ell_1) + \dots + c_n\varphi(\ell_n)$, where ℓ_1, \dots, ℓ_n are all the lattice points in Δ and c_1, \dots, c_n range over all values with $0 \leq c_1, \dots, c_n \leq 1$, $c_1 + \dots + c_n = 1$, and $c_1\ell_1 + \dots + c_n\ell_n = p$. This is also the height of the lower boundary of $\text{Conv}(S)$ over p , which is by definition equal to $\varphi'(p)$, so $f = \varphi'$ everywhere.

To prove the third statement, note that $\varphi' + \varphi_{\Delta}$ will be strictly convex piecewise linear on the intersection of the fan $\Sigma(\Delta)$ with the fan $\Sigma_{\varphi'}$ on

which φ' is strictly convex piecewise linear (the latter fan does not necessarily consist of strictly convex cones, but is still well defined). Thus we need to prove that for every cone C' in $\Sigma_{\varphi'}$ and C in $\Sigma(\Delta)$, $C \cap C'$ may be written as a cone over $\text{Conv}(v_1, \dots, v_k)$ with v_1, \dots, v_k lattice points in a common proper face of Δ . Because we have established that the graph of φ' over Δ is given by the lower faces of $\text{Conv}(S)$, $C' \cap \Delta$ is equal to the projection of a lower face of $\text{Conv}(S)$ to $Q_{\mathbb{R}}$, and since a lower face will be transverse to the kernel of the projection map, $C' \cap \Delta$ can be identified with the lower face P itself. P can be expressed as the union of the sets $\text{Conv}(0, f_i)$ where f_i ranges over all faces not containing the origin. (In general, if v is any point in a compact convex polytope, the polytope is equal to the union of $\text{Conv}(v, f_i)$ as f_i ranges over all faces not containing v .) The faces f_i can be written as the intersection of a proper unbounded face of $\text{Conv}(S)$ with P . The vertical (unbounded) faces of $\text{Conv}(S)$ are all of the form

$$\{(m, r) \mid m \in g, r \geq \varphi'(g)\}$$

for g a proper face of Δ . Thus the projection of each f_i to $Q_{\mathbb{R}}$ may be written as the convex hull of vertices in a common proper face of Δ , and we are done. \square

To construct the needed MPCP resolutions, we start by proving:

Proposition 5.9. *There exists an MPCP resolution $X'(\Delta_{P(2,4)}^*)$ of $X(\Delta_{P(2,4)}^*)$ such that the image of each cone in Σ' under g^* is a union of cones in the fan for $X'(\Delta_{P(2,4)}^*)$.*

Proof. By the case-by-case analysis in the proof of Proposition 5.7, we need to show that an MPCP resolution exists which, on the two-faces shown in Figures 3-5, is a refinement of the subdivisions shown. It is straightforward to check that there is a piecewise linear function $h : M_{\mathbb{R}} \rightarrow \mathbb{R}$ such that the vertices of the unbounded polytope $\tilde{\Delta}_{P(2,4)}^* \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$ defined by

$$\tilde{\Delta}_{P(2,4)}^* = \{(m, r) \mid r \geq h(m), m \in \Delta_{P(2,4)}^*\}$$

are as follows: the vertices $(v_i, 1)$, $1 \leq i \leq 6$, where again the v_i are the vertices of $\Delta_{P(2,4)}^*$; the vertices $(t_i, 1 - \epsilon)$, $1 \leq i \leq 4$ where the t_i are defined as in Proposition 5.7, the origin, and the vertex $((1, 1, -1, 1), 1 - \epsilon)$. Here ϵ is a small, positive rational number, and $(1, 1, -1, 1)$ is the vertex at the

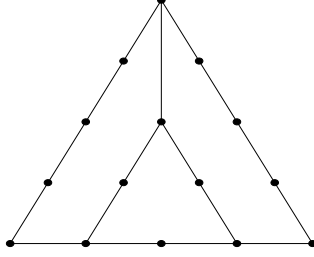


Figure 3: Subdivision of the faces $\text{Conv}(v_1, v_2, v_4)$ and $\text{Conv}(v_1, v_4, v_6)$.

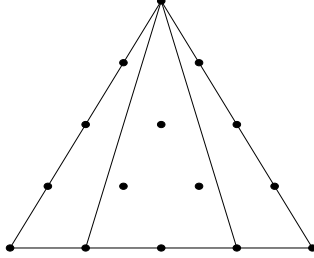


Figure 4: Subdivision of the faces $\text{Conv}(v_1, v_3, v_4)$ and $\text{Conv}(v_1, v_4, v_5)$.

center of the square S_1 . Then the restriction of h to the cones over the 2-faces of $\Delta_{P(2,4)}^*$ will be strictly convex piecewise linear on the subdivisions shown.

As demonstrated by Proposition 5.7, the images of cones in Σ'' are all cones contained in this subdivision. Thus, any MPCP resolution whose fan is a refinement of the fan on which h is strictly convex will suffice. We may construct such a refinement by the method of [4], altering h by small rational values at lattice points and scaling up until we get an MPCP resolution. The resulting function h' is guaranteed to be piecewise linear on $M_{\mathbb{R}}$ because, as is easy to check, h is strictly convex piecewise linear on a fan which is a subdivision of $\Sigma(\Delta_{P(2,4)}^*)$. \square

It remains to construct a compatible MPCP resolution of $X(\nabla)$. For this we will use the result of Lemma 5.8. Let h' be the MPCP function for $X(\Delta_{P(2,4)}^*)$ constructed by Proposition 5.9, and let φ_{∇} be the piecewise linear function on $M'_{\mathbb{R}}$ which is equal to 1 on the boundary of ∇ . Let j be the upper convex piecewise linear function on $M'_{\mathbb{R}}$ given by applying Lemma 5.8 to $h' \circ g^*$ and the reflexive polytope ∇ . Let $j' = j + \varphi_{\nabla}$. Then we have:

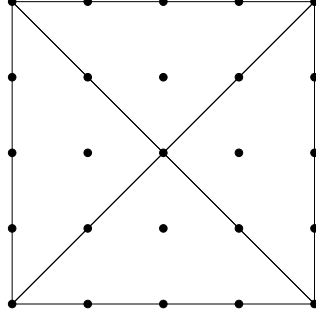


Figure 5: Subdivision of the face S_1 .

Proposition 5.10. *Let $\Sigma_{j'}$ be the fan on which j' is strictly convex piecewise linear, and let $\Sigma'_{j'}$ be the subfan consisting of all cones of $\Sigma_{j'}$ which are contained in some cone of Σ' . Then $\Sigma'_{j'}$ is a partial crepant resolution of Σ' and g^* induces a map from $\Sigma'_{j'}$ to $\Sigma_{h'}$, that is, if C is a cone in $\Sigma'_{j'}$ then $g^*(C)$ is contained in some cone of $\Sigma_{h'}$.*

Proof. As can be seen from the proof of Proposition 5.7, any cone of Σ' which is transverse to the kernel of g^* will be of the form $C(f)$ with f a 2-face of ∇ which is mapped injectively to a subset of a 2-face of $\Delta_{P(2,4)}^*$. The image of any such cone $C(f)$ will be a union of cones in the fan Σ_h on which the function h defined in Proposition 5.9 is strictly convex. Because h' is strictly convex on an MPCP subdivision of Σ_h , its restriction to $g^*(C(f))$ will be strictly convex on a lattice subdivision of $g^*(f)$. It follows that $h' \circ g^*$ will be strictly convex on an isomorphic lattice subdivision of f , and since φ_∇ is linear when restricted to f , so will $h' \circ g^* + \varphi_\nabla$. By the second property from Lemma 5.8, we therefore have that $j' = h' \circ g^* + \varphi_\nabla$ on $C(f)$, since both are upper convex piecewise linear on lattice subdivisions of f and are equal on lattice points. Thus the map g^* induces an isomorphism between the cones of $\Sigma'_{j'}$ contained in $C(f)$ and their images in $\Sigma_{h'}$.

For the remaining cones which are not mapped bijectively to their image under g^* , we use a case-by-case analysis. Using the same notation as Proposition 5.7, these are the cones over the faces e_1, e_2, e_3 and e_4 . A simple analysis shows that $h' \circ g^*$ must be piecewise linear on cones over subdivisions of these faces as shown in Figures 6-8. Since these are lattice subdivisions of the faces, we again conclude that by the second property of Lemma 5.8, $j' = h' \circ g^* + \varphi_\nabla$ on each such face. The cones over maximal faces in the

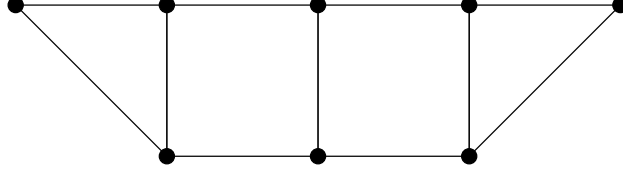


Figure 6: Induced subdivision of the face e_3 .

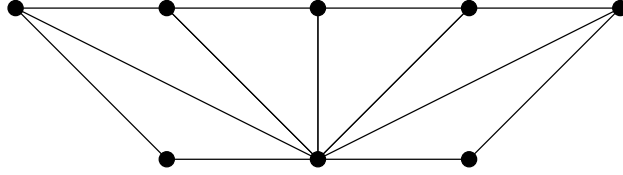


Figure 7: Induced subdivision of the face e_4 .

subdivisions will all be mapped to cones over line segments contained in the MPCP subdivision of $\Delta_{P(2,4)}^*$ corresponding to h' , so g^* will map these cones of $\Sigma'_{j'}$ to cones in $\Sigma_{h'}$. \square

By the third property of Lemma 5.8, j' is strictly convex piecewise linear on a partial crepant subdivision of ∇ , so we may use the GKZ procedure to produce an MPCP function which is piecewise linear on a subdivision of $\Sigma_{j'}$. The subfan of the subdivision consisting of cones contained in Σ' will be a subdivision of $\Sigma'_{j'}$, which we will call $\Sigma''_{j'}$. (It is evident from the proof of Proposition 5.10 that $\Sigma''_{j'}$ is a subdivision of Σ'' .) The map g^* will still induce a map between the fans $\Sigma''_{j'}$ and $\Sigma_{h'}$. Thus we have a map between the toric varieties $g^* : X(\Sigma''_{j'}) \rightarrow X(\Sigma_{h'})$, which also induces a map from each member of the family \mathcal{X}_{BB}^* to a member of \mathcal{X}_C^* . We can now formulate our main conjecture as:

Conjecture 5.11. *For a generic member $X \in \mathcal{X}_{BB}^*$, the map g^* from X to $g^*(X) \in \mathcal{X}_C^*$ is a small resolution of the singular Calabi-Yau variety $g^*(X)$.*

This will be addressed in a future paper.

Remark 5.12. As can be seen by analyzing the proof of Proposition 5.10, the subdivisions of the cones of Σ' in $\Sigma'_{j'}$ are already maximal except for the cones over the face

$$\text{Conv}((-1, -1, 3, -1, 0), (-1, -1, -1, 3, 0), (0, 0, 2, 0, -1), (0, 0, 0, 2, -1)).$$

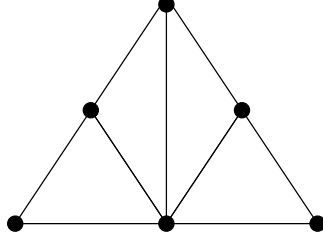


Figure 8: Induced subdivision of the faces e_1 and e_2 .

The two squares shown in Figure 6 represent open toric subvarieties of $X(\Sigma'_{j'})$ isomorphic to $A \times (\mathbb{C}^*)^2$, where A is the singular subvariety of $\text{Spec } \mathbb{C}[x, y, z, w]$ defined by $xy - zw = 0$, from the introduction. It can be shown that a generic member of the partially resolved family $\mathcal{X}_{BB}^* \subseteq X(\Sigma'_{j'})$ will intersect the singular locus of each open toric subvariety (corresponding to a square) transversely at one point, and thus acquire one conifold singularity on each square. As would be expected, these two conifold singularities map to two of the conifold singularities in the corresponding member of \mathcal{X}_C^* , which are located in the open toric subvarieties of $X(\Sigma_{h'})$ given by the cones in $N_{\mathbb{R}}$ over

$$\text{Conv}((-1, -1, 0, -1), (-1, -1, 1, -1))$$

and

$$\text{Conv}((-1, -1, 1, -1), (-1, -1, 2, -1)).$$

After resolution of $X(\Sigma'_{j'})$ to $X(\Sigma''_{j'})$, each square is subdivided into two triangles, constituting a small resolution of the two singularities of generic members of the family. Thus, to prove the conjecture, we would need to show that the map $g^* : X(\Sigma''_{j'}) \rightarrow X(\Sigma_{h'})$ induces a small resolution of the remaining two conifold singularities of generic members of \mathcal{X}_C^* , contained in the open toric subvarieties given by cones over

$$\text{Conv}((-1, -1, -1, -1), (-1, -1, 0, -1))$$

and

$$\text{Conv}((-1, -1, 2, -1), (-1, -1, 3, -1)).$$

It would then suffice to show that the map is a local analytic isomorphism at all other points.

6 Another example

To show that the same methods can yield results in other cases, we present a brief analysis of the example in section 3.3 in [7]. In this example, Morrison describes smoothing a Calabi-Yau hypersurface in the weighted projective space $\mathbb{P}^{(1,1,2,2,2,2)}$ by embedding $\mathbb{P}^{(1,1,2,2,2,2)}$ in \mathbb{P}^5 torically as the solution of a quadratic binomial equation. (Indeed, if z_0, \dots, z_5 are homogeneous coordinates on \mathbb{P}^5 , then the variety defined by $z_0 z_1 = z_2^2$ is isomorphic to $\mathbb{P}^{(1,1,2,2,2,2)}$.) If the quadratic binomial equation is deformed to a generic quadratic, the Calabi-Yau hypersurface will deform to a generic $(2, 4)$ complete intersection in \mathbb{P}^5 . Morrison then gives a birational morphism between the mirror of the $(2, 4)$ family and a subfamily of the mirror to the $\mathbb{P}^{(1,1,2,2,2,2)}$ family, which represents the mirror transition.

According to our approach, the mirror transition should be induced by the linear dual of the map that torically embeds $\mathbb{P}^{(1,1,2,2,2,2)}$ into \mathbb{P}^5 . A fan for $\mathbb{P}^{(1,1,2,2,2,2)}$ is contained in $M_{\mathbb{R}} \cong \mathbb{R}^4$ and given by the cones over the faces of the reflexive polytope P whose vertices are $(-1, -2, -2, -2)$ and f_1, f_2, f_3, f_4 (the standard basis). A toric embedding is given by the linear map $h : M_{\mathbb{R}} \rightarrow M'_{\mathbb{R}}$ which maps $\Sigma(P)$ to the fan $\Sigma(\Delta)$ for \mathbb{P}^5 (the same fan used previously), and has the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

The dual map $h^* : N'_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$ is given by the transpose. The lattice points contained in P are its vertices along with the point $(0, -1, -1, -1)$, so by Batyrev's construction the mirror family is given by (a Calabi-Yau compactification of)

$$-1 + a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_1^{-1} (X_2 X_3 X_4)^{-2} + a_6 (X_2 X_3 X_4)^{-1} = 0$$

in $T = \text{Spec } \mathbb{C}[M]$, where $X_i = z^{f_i}$. We identify the degenerate subfamily on the mirror as the subfamily satisfying $4a_1 a_5 = a_6^2$, and refer to it as \mathcal{X}_0^* . (This is the same as the condition $q_2 = 4$ in Morrison's description.) As usual, by using the torus action we may assume without loss that $a_1 = \dots = a_4 = 1$, and the degeneracy condition becomes $4a_5 = a_6^2$. We can factor to obtain

$$-1 + X_1(1 + (a_6/2)(X_1 X_2 X_3 X_4)^{-1})^2 + X_2 + X_3 + X_4 = 0. \quad (2)$$

For the mirror family to $(2, 4)$ complete intersections in \mathbb{P}^5 we again take a Calabi-Yau compactification of the equations

$$\begin{aligned} b_1 Y_1 + b_2 Y_2 + b_3 Y_3 + b_4 Y_4 &= 1 \\ b_5 Y_5 + b_6 (Y_1 Y_2 Y_3 Y_4 Y_5)^{-1} &= 1 \end{aligned}$$

in $T' = \text{Spec } \mathbb{C}[M']$, where $Y_i = z^{e_i}$, with e_1, \dots, e_5 the standard basis of M' . We can assume that all b_i except b_6 equal 1 by using the torus action (note that in the previous example, the same equations were used, but we chose to set all coefficients other than b_4 to 1, rather than b_6). Let us refer to this family with the single parameter b_6 as $\mathcal{X}_{(2,4)}^*$. Then we have:

Theorem 6.1. *After setting $b_6 = a_6/2$, the families $\mathcal{X}_{(2,4)}^*$ and \mathcal{X}_0^* are birationally equivalent via the map the map $f : T' \rightarrow T$ given by $X_1 = Y_4 Y_5^2$, $X_2 = Y_3$, $X_3 = Y_2$, $X_4 = Y_1$.*

Proof. Let f_{b_6} be the restriction of f to a single member of the family $\mathcal{X}_{(2,4)}^*$ with a particular value of $b_6 = a_6/2$. By using the factored form of the equation for \mathcal{X}_0^* (equation (2)), it is straightforward to check that the rational map from T to T' defined by

$$\begin{aligned} Y_1 &= X_4 \\ Y_2 &= X_3 \\ Y_3 &= X_2 \\ Y_4 &= 1 - X_2 - X_3 - X_4 \\ Y_5 &= (1 + b_6 (X_1 X_2 X_3 X_4)^{-1})^{-1} \end{aligned}$$

is a birational inverse for f_{b_6} . □

(Note that unlike in Theorem 4.1, the birational inverse depends on the parameter b_6 , essentially because both of the equations defining the family $\mathcal{X}_{(2,4)}^*$ must be used in solving for the inverse.) This confirms our prediction, since the map f is induced by the map $h^* : N'_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$. Like in the first example of quartic hypersurfaces in $P(2, 4)$, one may ask whether the map f can be extended to a larger domain, including more points of the ambient toric variety of the Batyrev-Borisov construction. We will not attempt to answer this question fully, but make a few observations. The ambient toric variety of $\mathcal{X}_{(2,4)}^*$ has a fan given by cones over the faces of the reflexive polytope ∇ , and $\nabla = \text{Conv}(\text{Newt}(\varphi_1), \text{Newt}(\varphi_2))$, where ∇ and the piecewise linear

functions φ_i have exactly the same definitions as before. Because $\varphi_1 \circ h^*$ is the piecewise linear function which equals one on the boundary of P , we have that $Newt(\varphi_1 \circ h^*) = P^*$. However, $\varphi_2 \circ h^*$ is a function equaling 2 on the vertices $(-1, -2, -2, -2)$ and $(1, 0, 0, 0)$ of P and zero on all other vertices. One can check that $Newt(\varphi_2 \circ h^*)$ is a polytope in $N_{\mathbb{R}}$ with vertices in the standard basis equal to the rows of

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}.$$

This polytope is not contained in P^* , so unlike in the previous example, we do not have that $h^*(\nabla) = P^*$. However, this does not necessarily prevent a similar approach from working. Although we will not attempt a full analysis, let us briefly examine what happens on the face

$$A = Conv((-1, 3, -1, -1), (-1, -1, 3, -1), (-1, -1, -1, 3)),$$

which is a location of singularities in generic members of \mathcal{X}_0^* (this is the face dual to the line segment $Conv((-1, -2, -2, -2), (1, 0, 0, 0))$ in P).

The two-faces of ∇ with images under h^* contained in $C(A)$ are:

$$\begin{aligned} k_1 &= Conv((0, 2, 0, 0, -1), (0, 0, 2, 0, -1), (0, 0, 0, 2, -1)) \\ k_2 &= Conv((-1, 3, -1, -1, 0), (-1, -1, 3, -1, 0), (0, 2, 0, 0, -1), (0, 0, 2, 0, -1)) \\ k_3 &= Conv((-1, 3, -1, -1, 0), (-1, -1, -1, 3, 0), (0, 2, 0, 0, -1), (0, 0, 0, 2, -1)) \\ k_4 &= Conv((-1, -1, 3, -1, 0), (-1, -1, -1, 3, 0), (0, 0, 2, 0, -1), (0, 0, 0, 2, -1)) \\ k_5 &= Conv((-1, 3, -1, -1, 0), (-1, -1, 3, -1, 0), (-1, -1, -1, 3, 0)). \end{aligned}$$

The face k_5 is of the type described in Proposition 5.6 and can be ignored. Figure 9 shows a possible MPCP subdivision of the face A along with a further subdivision which corresponds to the subsets $h^*(C_i) \cap A$, where C_i are cones in a compatible MPCP subdivision of the above faces of ∇ . The cone over the triangle which is the convex hull of the three lattice points in the relative interior of A is the image of $C(k_1)$, while the cones over the convex hulls of 1-faces of the center triangle and parallel 1-faces on the boundary are the images of $C(k_i)$ for $i = 2, 3, 4$. This shows that a map between MPCP

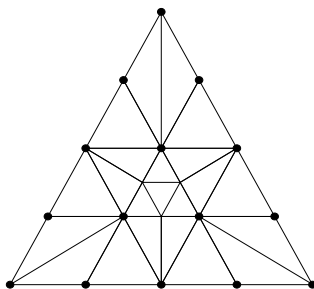


Figure 9: Subdivision of the face A .

resolutions of toric varieties containing the resolved families $\mathcal{X}_{P(2,4)}^*$ and \mathcal{X}_0^* is possible, at least on the part of the fan contained in $C(A)$.

Acknowledgements. I would like to thank the authors of the computer algebra program macaulay2 and its Polyhedra package. I would also like to thank Mark Gross for introducing me to the problem of mirror symmetry for complete intersections in Grassmannians.

References

- [1] V. V. Batyrev. Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties. *J. Alg. Geom.*, **3**:493–535, 1994. arXiv:alg-geom/9310003.
- [2] V. V. Batyrev and L. A. Borisov. On Calabi-Yau Complete Intersections in Toric Varieties. December 1994. arXiv:alg-geom/9412017.
- [3] V. V. Batyrev, I. Ciocan-Fontanine, B. Kim, and D. van Straten. Conifold Transitions and Mirror Symmetry for Calabi-Yau Complete Intersections in Grassmannians. October 1997. arXiv:alg-geom/9710022.
- [4] I.M. Gel’fand, A.V. Zelevinskii, and M.M. Kapranov. Equations of hypergeometric type and toric varieties. *Func. Anal. Appl.*, **23**(2):12–26, 1989. Engl. trans. 94-106.
- [5] M. Gross. Toric Degenerations and Batyrev-Borisov Duality. *Math. Ann.*, **333**:645–688, 2005. arXiv:math/0406171.

- [6] A. Mavlyutov. Degenerations and mirror contractions of Calabi-Yau complete intersections via Batyrev-Borisov mirror symmetry. 2011. arXiv:0910.0793v2.
- [7] David R. Morrison. Through the Looking Glass. In *Mirror Symmetry III*, pages 263–277. American Mathematical Society and International Press, 1999.
- [8] T. Oda. *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties*. Springer-Verlag, 1988.
- [9] M. Reid. The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible. *Math. Ann.*, **278**:329–334, 1987.